



# An Asymptotical Approximation of Outage Probability for Distributed MIMO Systems

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**Abstract:** The instantaneous capacity of multiple-input multiple-output (MIMO) systems over Rayleigh fading channel is known to follow a normal probability distribution when the number of antennas is large. In this paper, we extend this result for distributed MIMO (DMIMO) systems. First, we show that the instantaneous capacity of DMIMO systems is asymptotically equivalent to a Gaussian random variable, we then derive an asymptotical approximation of the outage probability which becomes tighter as the rate or the number of nodes and antennas of the system increases. This expression can be used to easily evaluate the outage probability of DMIMO systems, both for downlink and uplink cases. As an application, we utilise this expression to compare the theoretical throughput performance of MIMO and DMIMO systems.

**Keywords:** Cooperative communication, distributed MIMO, outage probability, asymptotical analysis, throughput.

## 1. Introduction

Calculation of the channel capacity for multiple-input multiple-output (MIMO) systems over the Rayleigh fading channel has attracted considerable research interests in the past decades [1–3]. Recently, it has been shown by various authors using different methods that the mutual information, or also known as instantaneous capacity, of the MIMO Rayleigh fading channel is equivalent to a Gaussian random variable (RV) [4–7]. This result has been theoretically obtained for large numbers of inputs and outputs of the MIMO channel and has been confirmed by simulations for even small numbers of inputs and outputs [5,6]. This result has been used to obtain an approximation of the outage probability for MIMO systems in [7].

In distributed MIMO (DMIMO) systems, several static nodes equipped with various numbers of antennas cooperate to communicate with a mobile node. This cooperation can be achieved in a multi-hop fashion, where one of the static node communicates with the mobile node using other static nodes to relay the information [8]; or in a distributed manner, where each static node shares the same data via a wired or wireless link and all the static nodes communicate together with the mobile node [9–11]. The latter scenario is considered here.

In this paper, we propose an asymptotical approximation of the outage probability for DMIMO systems, relying on the system model introduced in Section 2 [9]. In Section 3, we prove that the instantaneous capacity of DMIMO systems over Rayleigh fading channel is asymptotically equivalent to a Gaussian RV by using the replica method [12], and the multiple saddle point integration technique [13]. First, we derive the moment generating function of the instantaneous capacity of the uplink for a large number of nodes and antennas. We show that its formulation is equivalent to the formulation of the moment generating function of a Gaussian RV. We then obtain the mean and variance of the instantaneous capacity by matching them with those of a Gaussian RV

and use them to express our asymptotical approximation of the outage probability for the downlink and uplink of DMIMO systems. Numerical results in Section 4 indicate that our asymptotical approximation of the outage probability becomes tighter as the rate or the number of nodes and antennas of the system increases. In Section 4, we introduce some applications for our results, e.g., we utilise our expression to show the throughput gain that DMIMO systems can achieve over MIMO systems when an unlimited number of automatic retransmission requests (ARQs) is assumed. Finally, conclusions are drawn in Section 5.

## 2. System Model for DMIMO systems

We consider a DMIMO communication system composed of  $m + 1$  nodes, where  $m$  base stations (BSs) equipped with  $p$  antennas cooperate to transmit/receive data to/from a mobile station (MS) equipped with  $q$  antennas. The  $m$  BSs are in different locations and therefore the links between each of the  $m$  BSs and the MS have different path losses [9]. As a result, the receive signal can be expressed as

$$\mathbf{r} = \sqrt{\epsilon/n}\tilde{\mathbf{H}}\mathbf{s} + \mathbf{n}, \quad (1)$$

where  $\epsilon$  is the average transmitted signal energy,  $\mathbf{s} \in \mathbb{C}^{N_t}$  is a normalised to unit-power transmitted signal,  $\tilde{\mathbf{H}} \in \mathbb{C}^{N_r \times N_t}$  characterises the propagation variation of each link between any nodes, and  $\mathbf{n} \in \mathbb{C}^{N_r}$  is a vector of independent zero-mean complex Gaussian noise entries with a double-sided variance of  $N_0/2$ . The total number of transmit and receive antennas of the DMIMO system is defined as  $N_t$  and  $N_r$ , respectively, and  $n$  is the number of transmit antenna per node. In the uplink case,  $N_t = n = q$  and  $N_r = mp$ , while in the downlink case,  $N_t = mp$ ,  $N_r = q$  and  $n = p$ . The matrix  $\tilde{\mathbf{H}}$  is defined as the element-wise product between the matrix  $\mathbf{\Sigma} \in \mathbb{R}_+^{N_r \times N_t}$  that accounts for the average path loss and shadowing, and the matrix  $\mathbf{H} \in \mathbb{C}^{N_r \times N_t}$  that characterises the MIMO Rayleigh fading channel such that  $\tilde{\mathbf{H}} = \mathbf{\Sigma} \odot \mathbf{H}$ , where  $\odot$  is the element-wise product between any two matrices and  $\mathbb{R}_+ = \{x \in \mathbb{R} | x \geq 0\}$ . The average path loss and shadowing is different for each of the  $m$  BSs, thus  $\mathbf{\Sigma} = \mathbf{\Gamma} \triangleq [\mathbf{\Gamma}_1^\dagger, \mathbf{\Gamma}_2^\dagger, \dots, \mathbf{\Gamma}_m^\dagger]^\dagger$  and  $\mathbf{\Sigma} = \mathbf{\Gamma}^\dagger$  in the uplink and downlink cases, respectively, with  $\mathbf{\Gamma}_i \in \mathbb{R}_+^{q \times p}$ . Moreover, we consider that  $\mathbf{H}$  is a random matrix having zero-mean and variance of 0.5 i.i.d. Gaussian entries, and we assume that  $\mathbf{H}$  remains fixed for the whole transmission when it has been chosen. Thus, the channel is non-ergodic and the outage probability, i.e., the probability that the transmission rate  $R$  exceeds the instantaneous capacity of the aggregate channel  $C(\tilde{\mathbf{H}}) \triangleq \lambda \ln \phi(\tilde{\mathbf{H}})$ , is defined as

$$P_{\text{out}} \triangleq \mathbf{P}(C(\tilde{\mathbf{H}}) < R), \quad (2)$$

where  $\phi(\tilde{\mathbf{H}}) = |\mathbf{I}_{N_r} + \gamma \tilde{\mathbf{H}}\tilde{\mathbf{H}}^\dagger/n|$ ,  $\gamma = \epsilon/N_0$  is the average signal-to-noise-ratio (SNR),  $\mathbf{I}_{N_r}$  is a  $N_r \times N_r$  identity matrix,  $\tilde{\mathbf{H}}^\dagger$  indicates the Hermitian transpose of  $\tilde{\mathbf{H}}$ , and  $\lambda = 1$  if the capacity is expressed in nats/s/Hz or  $\lambda = \frac{1}{\ln(2)}$  if the capacity is expressed in bits/s/Hz.

Recently, it has been shown in [4–7] that the instantaneous capacity  $C(\mathbf{H})$  of MIMO systems over Rayleigh fading channel, i.e., for  $m = 1$ , tends to be a Gaussian RV as  $p$  and  $q$  grows to infinity. In the next section, we extend this result for DMIMO systems over Rayleigh fading channel, i.e., for  $m > 1$ , and show that  $C(\tilde{\mathbf{H}})$  is asymptotically equivalent to a Gaussian RV  $Z$  with moment generating function defined by

$$M_Z(t) \triangleq \mathbf{E}\{\exp(tZ)\} = \exp(t\mu_z + t^2\sigma_z^2/2), \quad (3)$$

when  $m$ ,  $p$  and  $q$  grows to infinity, and where  $\mu_z \in \mathbb{R}$  and  $\sigma_z^2 > 0$ .

### 3. Asymptotical approximation of the DMIMO outage probability

Let us re-express  $\phi(\tilde{\mathbf{H}})$ , which is given below (2), as follows

$$\phi(\tilde{\mathbf{H}}) = \left| \begin{array}{cc} \mathbf{I}_{N_r} & j\sqrt{\gamma/n}\tilde{\mathbf{H}} \\ j\sqrt{\gamma/n}\tilde{\mathbf{H}}^\dagger & \mathbf{I}_{N_t} \end{array} \right|. \quad (4)$$

Applying Theorem 1 in the Appendix to  $\mathbf{A} = \left( \mathbf{I}_{N_r} + \gamma\tilde{\mathbf{H}}\tilde{\mathbf{H}}^\dagger/n \right)$  and using (4), we get

$$\phi(\tilde{\mathbf{H}})^{-1} = \int_{\mathbb{C}^{N_r}} \int_{\mathbb{C}^{N_t}} \exp \left( -\pi \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix}^\dagger \begin{bmatrix} \mathbf{I}_{N_r} & j\sqrt{\gamma/n}\tilde{\mathbf{H}} \\ j\sqrt{\gamma/n}\tilde{\mathbf{H}}^\dagger & \mathbf{I}_{N_t} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix} \right) d\mathbf{x}d\mathbf{y}, \quad (5)$$

where  $\mathbf{x} \in \mathbb{C}^{N_r}$  and  $\mathbf{y} \in \mathbb{C}^{N_t}$ . At this stage, replicating  $u$  times the integral in (5) with several variables [12], we obtain  $\phi(\tilde{\mathbf{H}})^{-u} = \exp(-u/\lambda)C(\tilde{\mathbf{H}}) =$

$$\int_{\mathbb{C}^{N_r \times u}} \int_{\mathbb{C}^{N_t \times u}} \exp \left( -\pi \operatorname{tr} \left[ \left( \mathbf{X}^\dagger \mathbf{X} + \mathbf{Y}^\dagger \mathbf{Y} \right) + j\sqrt{\gamma/n} \left( \tilde{\mathbf{H}} \mathbf{Y} \mathbf{X}^\dagger + \mathbf{X} \mathbf{Y}^\dagger \tilde{\mathbf{H}}^\dagger \right) \right] \right) d\mathbf{X}d\mathbf{Y}, \quad (6)$$

where  $\operatorname{tr}[\cdot]$  denotes the trace of a matrix,  $\mathbf{X} = [\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_u]$ ,  $\mathbf{Y} = [\mathbf{y}_1, \mathbf{y}_2, \dots, \mathbf{y}_u]$ ,  $\mathbf{X} \in \mathbb{C}^{N_r \times u}$ , and  $\mathbf{Y} \in \mathbb{C}^{N_t \times u}$ . Averaging (6) with respect to  $\mathbf{H}$ ,  $\mathbf{E}\{\exp[(-u/\lambda)C(\tilde{\mathbf{H}})]\} =$

$$M_{C(\tilde{\mathbf{H}})}(-u/\lambda) = \pi^{-N_r N_t} \int_{\mathbb{C}^{N_r \times N_t}} \int_{\mathbb{C}^{N_r \times u}} \int_{\mathbb{C}^{N_t \times u}} \exp(-\pi \operatorname{tr}[(\mathbf{X}^\dagger \mathbf{X} + \mathbf{Y}^\dagger \mathbf{Y})]) \psi(\mathbf{H}) d\mathbf{H}d\mathbf{X}d\mathbf{Y}, \quad (7)$$

where  $\psi(\mathbf{H}) = \exp\left(-\operatorname{tr}\left[j\left(\tilde{\mathbf{H}}\mathbf{B}^\dagger + \mathbf{B}\tilde{\mathbf{H}}^\dagger\right) - \mathbf{H}\mathbf{H}^\dagger\right]\right)$ , and  $\mathbf{B} \triangleq \pi\sqrt{\gamma/n}\mathbf{X}\mathbf{Y}^\dagger$ ,  $\mathbf{B} \in \mathbb{C}^{N_r \times N_t}$ .

Integrating  $\psi(\mathbf{H})$  in (7) with respect to  $\mathbf{H}$ , we can rewrite (7) as  $M_{C(\tilde{\mathbf{H}})}(-u/\lambda) =$

$$\int_{\mathbb{C}^{N_r \times u}} \int_{\mathbb{C}^{N_t \times u}} \exp \left( -\pi \operatorname{tr} \left[ \left( \mathbf{X}^\dagger \mathbf{X} + \mathbf{Y}^\dagger \mathbf{Y} \right) - \pi^2 \frac{\gamma}{n} (\boldsymbol{\Sigma} \odot \mathbf{X} \mathbf{Y}^\dagger) (\boldsymbol{\Sigma} \odot \mathbf{X} \mathbf{Y}^\dagger)^\dagger \right] \right) d\mathbf{X}d\mathbf{Y}. \quad (8)$$

Let  $\boldsymbol{\Gamma}_i = \sqrt{\gamma_i} \mathbf{1}^{v \times p}$ , where  $\gamma_i \in \mathbb{R}_+$  is the path loss coefficient of the  $i$ -th BS-MS link,  $\mathbf{1}$  is a  $v \times p$  matrix with all its elements equal to 1, and  $v$  is a dummy variable. Then the following equalities  $(\boldsymbol{\Gamma} \odot \mathbf{X} \mathbf{Y}^\dagger) = (\boldsymbol{\Gamma} \odot \mathbf{X}) \mathbf{Y}^\dagger$  and  $(\boldsymbol{\Gamma} \odot \mathbf{X} \mathbf{Y}^\dagger) = \mathbf{X} (\boldsymbol{\Gamma}^\dagger \odot \mathbf{Y}^\dagger)$  hold in the uplink and downlink, respectively, with  $v = q$  in the left hand side and  $v = u$  in the right hand side of each equality. Consequently, (8) can be re-expressed in the uplink as follows

$$M_{C(\tilde{\mathbf{H}})}(-u/\lambda) = \int_{\mathbb{C}^{N_r \times u}} \int_{\mathbb{C}^{N_t \times u}} \exp \left( -\pi \operatorname{tr} \left[ \left( \mathbf{X}^\dagger \mathbf{X} + \mathbf{Y}^\dagger \mathbf{Y} \right) - \pi^2 \frac{\gamma}{n} \mathbf{W}^\dagger \mathbf{W} \mathbf{Y}^\dagger \mathbf{Y} \right] \right) d\mathbf{X}d\mathbf{Y}, \quad (9)$$

where  $\mathbf{W} \triangleq (\boldsymbol{\Gamma} \odot \mathbf{X})$  and  $\operatorname{tr}[\mathbf{W} \mathbf{Y}^\dagger \mathbf{Y} \mathbf{W}^\dagger] = \operatorname{tr}[\mathbf{W}^\dagger \mathbf{W} \mathbf{Y}^\dagger \mathbf{Y}]$ . The term  $\exp(\operatorname{tr}[-\pi^2 \frac{\gamma}{n} \mathbf{W}^\dagger \mathbf{W} \times \mathbf{Y}^\dagger \mathbf{Y}])$  in (9) can then be re-expressed using Equality 1 in the Appendix as

$$\prod_{a=1}^u \prod_{b=1}^u \frac{n}{j2\pi} \int_{\mathcal{D}_{\mathbf{D}_{b,a}}^j} \int_{\mathcal{D}_{\mathbf{G}_{a,b}}} \exp \left( \mathbf{D}_{b,a} [n \mathbf{G}_{a,b} - \pi \sqrt{\gamma} (\mathbf{W}^\dagger \mathbf{W})_{a,b}] - \pi \sqrt{\gamma} \mathbf{G}_{a,b} (\mathbf{Y}^\dagger \mathbf{Y})_{b,a} \right) d\mathbf{D}_{b,a} d\mathbf{G}_{a,b}. \quad (10)$$

Inserting (10) into (9) and followed by straightforward simplifications, we obtain

$$M_{C(\tilde{\mathbf{H}})}(-u/\lambda) = \left( \frac{n}{j2\pi} \right)^{u^2} \int_{\mathcal{D}_d^j} \int_{\mathcal{D}_g} \exp(n \operatorname{tr}[\mathbf{D}\mathbf{G}]) \int_{\mathbb{C}^{N_r \times u}} \exp(-\pi \operatorname{tr}[\mathbf{X}^\dagger \mathbf{X} + \sqrt{\gamma} \mathbf{D} \mathbf{W}^\dagger \mathbf{W}]) d\mathbf{X} \\ \times \int_{\mathbb{C}^{N_t \times u}} \exp(-\pi \operatorname{tr}[\mathbf{Y}^\dagger \mathbf{Y} (\mathbf{I}_u + \sqrt{\gamma} \mathbf{G})]) d\mathbf{Y} d\mathbf{D} d\mathbf{G}, \quad (11)$$

with  $\mathcal{D}_d^j \triangleq \mathbf{D}_0 + (j\mathbb{R}^{u \times u})$  and  $\mathcal{D}_g \triangleq \mathbf{G}_0 + (\mathbb{R}^{u \times u})$ ,  $\mathbf{D}_0, \mathbf{G}_0 \in \mathbb{C}^{u \times u}$ . Notice that  $\text{tr}[\mathbf{Y}^\dagger \mathbf{Y}(\mathbf{I}_u + \sqrt{\gamma} \mathbf{G})] = \sum_{i=1}^{N_t} \mathbf{Y}_i(\mathbf{I}_u + \sqrt{\gamma} \mathbf{G}) \mathbf{Y}_i^\dagger$  and  $\text{tr}[\mathbf{X}^\dagger \mathbf{X} + \sqrt{\gamma} \mathbf{D} \mathbf{W}^\dagger \mathbf{W}] = \sum_{i=1}^{N_r} \mathbf{X}_i(\mathbf{I}_u + \sqrt{\gamma} \gamma_{\lfloor i/p \rfloor} \mathbf{D}) \mathbf{X}_i^\dagger$  in (11), where  $\mathbf{X}_i$  and  $\mathbf{Y}_i$  are the  $i$ -th row vectors of  $\mathbf{X}$  and  $\mathbf{Y}$ , respectively, and  $\lfloor \cdot \rfloor$  is the integer floor operator. Then, applying Theorem 1 in the Appendix for integrating the integrand with respect to  $\mathbf{X}$  and  $\mathbf{Y}$  in (11),  $M_{C(\tilde{\mathbf{H}})}(-u/\lambda) =$

$$\exp \left( un[(m\alpha + \beta) \ln(\omega) + \alpha \sum_{i=1}^m \ln(v_i)] \right) \left( \frac{n}{j2\pi} \right)^{u^2} \int_{\mathcal{D}_d^j} \int_{\mathcal{D}_g} \exp(n\varphi(\mathbf{D}, \mathbf{G})) d\mathbf{D} d\mathbf{G}, \quad (12)$$

where  $\alpha \triangleq p/n$ ,  $\beta \triangleq q/n$ ,  $v_i \triangleq 1/\gamma_i$ ,  $\omega \triangleq 1/\sqrt{\gamma}$ , and  $\varphi(\mathbf{D}, \mathbf{G})$  is defined as

$$\varphi(\mathbf{D}, \mathbf{G}) \triangleq \text{tr}(\mathbf{D}\mathbf{G}) - \alpha \sum_{i=1}^m \ln \det(v_i \omega \mathbf{I}_u + \mathbf{D}) - \beta \ln \det(\omega \mathbf{I}_u + \mathbf{G}). \quad (13)$$

In order to asymptotically compute the integral in (12), i.e., for  $m, p, q \rightarrow \infty$ , we apply the multidimensional saddle point integration method [13]. First, we expand the Taylor series of  $\varphi(\mathbf{D}, \mathbf{G})$  for  $\mathbf{D} = \mathbf{D}_0 + \delta\mathbf{D}$  and  $\mathbf{G} = \mathbf{G}_0 + \delta\mathbf{G}$  to its second order, and we obtain

$$\begin{aligned} \varphi(\mathbf{D}, \mathbf{G}) &\approx \varphi(\mathbf{D}_0, \mathbf{G}_0) + \text{tr}[(\mathbf{G}_0 - \alpha \sum_{i=1}^m (v_i \omega \mathbf{I}_u + \mathbf{D}_0)^{-1}) \delta\mathbf{D} + (\mathbf{D}_0 - \beta(\omega \mathbf{I}_u + \mathbf{G}_0)^{-1}) \delta\mathbf{G} \\ &\quad + \frac{1}{2} (\alpha \sum_{i=1}^m (v_i \omega \mathbf{I}_u + \mathbf{D}_0)^{-2} (\delta\mathbf{D})^2 + 2\delta\mathbf{D}\delta\mathbf{G} + \beta(\omega \mathbf{I}_u + \mathbf{G}_0)^{-2} (\delta\mathbf{G})^2)]. \end{aligned} \quad (14)$$

According to (14), we obtain two critical points as follows

$$\mathbf{G}_0 = \alpha \sum_{i=1}^m (v_i \omega \mathbf{I}_u + \mathbf{D}_0)^{-1}, \mathbf{D}_0 = \beta(\omega \mathbf{I}_u + \mathbf{G}_0)^{-1}. \quad (15)$$

The matrices  $\mathbf{D}_0$  and  $\mathbf{G}_0$  are invariant in the replica space and thus are proportional to the identity matrix  $\mathbf{I}_u$  [5]. Consequently, we set  $\mathbf{D}_0 = \bar{d}\mathbf{I}_u$  and  $\mathbf{G}_0 = \bar{g}\mathbf{I}_u$  and assume that the absolute maximum of  $\Re\varphi(\mathbf{D}, \mathbf{G})$  occurs for these values of  $\mathbf{D}_0$  and  $\mathbf{G}_0$ . Inserting the first equation into the second equation of (15) and considering  $\mathbf{D}_0 = d\mathbf{I}_u$ , we obtain the following polynomial equation

$$P(d) = (d\omega - \beta) \prod_{i=1}^m (v_i \omega + d) + d\alpha \sum_{i=1}^m \prod_{\substack{k=1 \\ k \neq i}}^m (v_k \omega + d) = 0. \quad (16)$$

We observe for different values of  $m$  that only one root amongst the  $m+1$  of  $P(d)$  is real and nonnegative. We denote this root as  $d_0$ , we then set  $\mathbf{D}_0 = d_0 \mathbf{I}_u$  and obtain  $\mathbf{G}_0$  using the first equation of (15). Applying  $\mathbf{D}_0$  and  $\mathbf{G}_0$  in (12), we asymptotically express  $M_{C(\tilde{\mathbf{H}})}(-u/\lambda)$  as  $M_{C(\tilde{\mathbf{H}})}(-u/\lambda) \rightarrow \exp(un[(m\alpha + \beta) \ln(\omega) + \alpha \sum_{i=1}^m \ln(v_i)] + n\varphi(\mathbf{D}_0, \mathbf{G}_0))$

$$\times \left( \frac{n}{j2\pi} \right)^{u^2} \int_{\mathcal{D}_d^j} \int_{\mathcal{D}_g} \exp \left( \frac{n}{2} \text{tr} \left[ \hat{d}_0 (\delta\mathbf{D})^2 + 2\delta\mathbf{D}\delta\mathbf{G} + \hat{g}_0 (\delta\mathbf{G})^2 \right] \right) d(\delta\mathbf{D}) d(\delta\mathbf{G}), \quad (17)$$

where  $\hat{d}_0 = \alpha \sum_{i=1}^m (v_i \omega + d_0)^{-2}$ ,  $\hat{g}_0 = d_0^2/\beta$ . Integrating (17) along the steepest descent paths for  $\delta\mathbf{D}$  and  $\delta\mathbf{G}$ , the integral part in (17) is equal to  $(1 - \hat{d}_0 \hat{g}_0)^{-u^2/2}$  and we have

$$\begin{aligned} M_{C(\tilde{\mathbf{H}})}(-u/\lambda) &\rightarrow \exp \left( un \left[ (m\alpha + \beta) \ln(\omega) + \alpha \sum_{i=1}^m \ln \left( \frac{v_i}{v_i \omega + d_0} \right) + \beta \ln \left( \frac{d_0}{\beta} \right) \right. \right. \\ &\quad \left. \left. + \alpha \sum_{i=1}^m \frac{d_0}{v_i \omega + d_0} \right] + \frac{u^2}{2} \left[ -\ln(1 - \hat{d}_0 \hat{g}_0) \right] \right). \end{aligned} \quad (18)$$

Equation (18) shows that  $M_{C(\tilde{\mathbf{H}})}(-u/\lambda)$  can be expressed as  $M_Z(t)$  in (3) for  $Z = C(\tilde{\mathbf{H}})$ ,  $t = -u/\lambda$ , and

$$\begin{cases} \mu_Z = -\lambda n \left[ \ln(\omega^{(m\alpha+\beta)}) + \alpha \sum_{i=1}^m \ln\left(\frac{v_i}{\omega v_i + d_0}\right) + \beta \ln\left(\frac{d_0}{\beta}\right) + \alpha \sum_{i=1}^m \frac{d_0}{v_i \omega + d_0} \right], \\ \sigma_Z^2 = -\lambda^2 \ln\left(1 - \frac{\alpha}{\beta} \sum_{i=1}^m \left(\frac{d_0}{d_0 + \omega v_i}\right)^2\right). \end{cases} \quad (19)$$

Therefore  $C(\tilde{\mathbf{H}})$  is asymptotically equivalent to a Gaussian RV  $Z$ , with  $\mu_{C(\tilde{\mathbf{H}})} = \mu_Z$  and  $\sigma_{C(\tilde{\mathbf{H}})}^2 = \sigma_Z^2$ , respectively, in (19). This result is valid if  $u$  can be extended from the positive integers to the whole complex plane  $\Re u > 0$ . Following the replica method [12], we assume as in [5–7] that (18) can be extended to a compact neighborhood of  $u = 0^+$ . Finally,  $C(\tilde{\mathbf{H}})$  being asymptotically equivalent to a Gaussian RV and with (2), the outage probability of DMIMO systems can be well approximated for large values of  $m$ ,  $p$ , and  $q$  by

$$P_{\text{out}} \approx Q\left([\mu_{C(\tilde{\mathbf{H}})} - R]/\sigma_{C(\tilde{\mathbf{H}})}\right). \quad (20)$$

Notice that the derivation of the outage probability from (9) to (20) has been undertaken for the uplink case, however its generalisation for the downlink case is straightforward. Knowing that  $(\mathbf{\Sigma} \odot \mathbf{X}\mathbf{Y}^\dagger) = \mathbf{X}(\mathbf{\Sigma}^\dagger \odot \mathbf{Y}^\dagger)$ , (8) can be re-expressed in the downlink case as

$$M_{C(\tilde{\mathbf{H}})}(-u/\lambda) = \int_{\mathbb{C}^{N_r \times u}} \int_{\mathbb{C}^{N_t \times u}} \exp\left\{-\pi \operatorname{tr}\left[(\mathbf{X}^\dagger \mathbf{X} + \mathbf{Y}^\dagger \mathbf{Y}) - \pi^2 \frac{\gamma}{n} \mathbf{X}\mathbf{W}^\dagger \mathbf{W}\mathbf{X}^\dagger\right]\right\} d\mathbf{X} d\mathbf{Y}, \quad (21)$$

where  $\mathbf{W}^\dagger \triangleq (\mathbf{\Sigma}^\dagger \odot \mathbf{Y}^\dagger)$ . Following the different steps of the derivation from (10) to (20) with (21) instead of (9) as a starting point, the outage probability can be expressed as in (20) with  $\mu_{C(\tilde{\mathbf{H}})}$  and  $\sigma_{C(\tilde{\mathbf{H}})}$  given in (19). However, in the downlink case we have  $n = p$ ,  $\alpha = 1$  and  $\beta = q/p$  in (19), instead of  $n = q$ ,  $\alpha = p/q$  and  $\beta = 1$  in the uplink case.

## 4. Numerical results and Applications for $P_{\text{out}}$

### 4.1 Accuracy of $P_{\text{out}}$

The accuracy of our asymptotical approximation in (20) depends on the values of  $m$ ,  $p$  and  $q$ . In Table 1, we establish numerically for a MIMO system, i.e.,  $m = 1$ , and a DMIMO system with two BSs, i.e.,  $m = 2$ , respectively, the values of  $p$  and  $q$  for which our asymptotical approximation in (20) has an sufficient accuracy. We define  $\varepsilon_\gamma(\text{dB}) \triangleq |\gamma_{TH}(\text{dB}) - \gamma_{MC}(\text{dB})|$  as the accuracy metric, where  $\gamma_{TH}$  and  $\gamma_{MC}$  are the theoretical SNR and the SNR obtained via Monte-Carlo simulation, respectively, which are required to achieve a certain rate  $R$  for a given outage probability  $P_{\text{out}}$ . Thus, the lower is  $\varepsilon_\gamma(\text{dB})$ , the more accurate is our expression. We consider that  $\varepsilon_\gamma < 0.1$  dB is an sufficient accuracy, since if  $\varepsilon_\gamma < 0.1$  dB then  $\gamma_{TH}$  and  $\gamma_{MC}$  differ by less than 3%.

In the case of DMIMO system with two BSs, we define  $\Delta_\gamma(\text{dB}) \triangleq \gamma_2(\text{dB}) - \gamma_1(\text{dB})$  as the SNR offset between the two BSs resulting from unequal path losses and shadowing between each BS and the MS.

The condition  $\mu_z \in \mathbb{R}$  in (19) holds only if  $d_0 \geq 0$ . Therefore, the root  $d_0$  of  $P(d)$  in (16) must be carefully selected amongst  $m + 1$  roots, in order to evaluate (20). This

Table 1: Accuracy  $\varepsilon_\gamma(\text{dB})$  of our asymptotical approximation in (20) for a MIMO system ( $m = 1$ ), and a DMIMO system with two base stations ( $m = 2$ )

$p$	$m = 1$									$m = 2$							
	$R = 1$			$P_{\text{out}} = 0.01$						$\Delta_\gamma = 0\text{dB}$			$R = 4$				
	$\beta = 1$						$R = 1$										
	$P_{\text{out}}$			$R$			$\beta$			$R$			$\Delta_\gamma(\text{dB})$				
	0.001	0.01	0.1	1	4	10	1	2	3	1	4	10	-10	-5	0	5	10
1	1.01	0.50	0.47	0.50	2.88	9.45	0.50	0.67	0.31	0.65	1.49	1.55	0.98	1.33	1.49	1.33	0.97
2	1.43	0.80	<b>0.010</b>	0.80	0.43	0.25	0.80	0.47	0.31	0.48	<b>0.07</b>	<b>0.04</b>	0.13	0.11	<b>0.07</b>	0.11	0.12
3	1.05	0.53	<b>0.060</b>	0.53	0.20	0.20	0.53	0.29	0.20	0.29	<b>0.06</b>	<b>0.03</b>	0.13	<b>0.10</b>	<b>0.06</b>	<b>0.09</b>	0.12
4	0.72	0.35	<b>0.046</b>	0.35	0.15	0.11	0.35	0.18	0.12	0.18	<b>0.06</b>	<b>0.02</b>	0.12	<b>0.09</b>	<b>0.06</b>	<b>0.09</b>	0.12
5	0.50	0.24	<b>0.032</b>	0.24	0.11	<b>0.07</b>	0.24	0.12	<b>0.08</b>	0.12	<b>0.05</b>	<b>0.02</b>	<b>0.09</b>	<b>0.07</b>	<b>0.05</b>	<b>0.07</b>	<b>0.09</b>
6	0.36	0.17	<b>0.023</b>	0.17	<b>0.08</b>	<b>0.05</b>	0.17	<b>0.09</b>	<b>0.06</b>	<b>0.09</b>	<b>0.04</b>	<b>0.01</b>	<b>0.07</b>	<b>0.06</b>	<b>0.04</b>	<b>0.06</b>	<b>0.08</b>
7	0.27	0.13	<b>0.018</b>	0.13	<b>0.07</b>	<b>0.04</b>	0.13	<b>0.07</b>	<b>0.04</b>	<b>0.06</b>	<b>0.04</b>	<b>0.01</b>	<b>0.06</b>	<b>0.05</b>	<b>0.04</b>	<b>0.05</b>	<b>0.06</b>
8	0.21	<b>0.10</b>	<b>0.014</b>	<b>0.10</b>	<b>0.06</b>	<b>0.03</b>	<b>0.10</b>	<b>0.05</b>	<b>0.03</b>	<b>0.05</b>	<b>0.03</b>	<b>0.01</b>	<b>0.05</b>	<b>0.04</b>	<b>0.03</b>	<b>0.04</b>	<b>0.05</b>
9	0.17	<b>0.08</b>	<b>0.012</b>	<b>0.08</b>	<b>0.05</b>	<b>0.03</b>	<b>0.08</b>	<b>0.04</b>	<b>0.03</b>	<b>0.04</b>	<b>0.02</b>	<b>0.01</b>	<b>0.04</b>	<b>0.03</b>	<b>0.02</b>	<b>0.03</b>	<b>0.04</b>
10	0.14	<b>0.06</b>	<b>0.009</b>	<b>0.06</b>	<b>0.04</b>	<b>0.02</b>	<b>0.06</b>	<b>0.03</b>	<b>0.02</b>	<b>0.03</b>	<b>0.02</b>	<b>0.01</b>	<b>0.03</b>	<b>0.03</b>	<b>0.02</b>	<b>0.03</b>	<b>0.03</b>

selection is straightforward for the case of  $m = 1$  [7]. However, for the case of  $m > 1$ , the selection of  $d_0$  is a problem by itself which has been solved in [14]. The expression of  $d_0$  for  $m = 2$  can also be found in [14].

In Table 1, various  $\varepsilon_\gamma(\text{dB})$  values obtained by using either a MIMO system, i.e.,  $m = 1$ , or a DMIMO system, i.e.,  $m = 2$ , are displayed for different  $P_{\text{out}}$ ,  $R$ ,  $\beta$  and  $\Delta_\gamma(\text{dB})$  values against the number of antennas  $p$ , in the downlink case. We consider that a sufficient accuracy is reached when  $\Delta_\gamma < 0.1$  dB and the corresponding  $\Delta_\gamma$  values are highlighted in the table. In the case of  $m = 1$ , going from left to right, the first, second, and third block of three columns indicates that the accuracy of (20) increases as  $P_{\text{out}}$ ,  $\beta$ ,  $R$ , increases, respectively. More generally, these results shows that  $\varepsilon_\gamma(\text{dB})$  decreases as  $p$  increases, it implies that the accuracy of (20) increases with the number of antennas of the MIMO system. This result is in line with the results obtained in [5, 6] for  $m = 1$ . The results also indicate that (20) is not very accurate for low values of  $p$ ,  $\beta$ ,  $P_{\text{out}}$  and  $R$ . In the case of  $m = 2$ , the accuracy of (20) also increases as  $R$  increases, and decreases as  $|\Delta_\gamma|$  increases. Moreover, the results pinpoint that (20) is accurate for a wider range of  $p$  values than in the MIMO case, when the same  $R$  and  $\beta$  values are considered. This result was expected since when  $m$  increases the overall number of antennas in the system increases.

#### 4.2 Applications for $P_{\text{out}}$

Our asymptotical approximation in (20) can be used to evaluate and compare the outage probability of DMIMO systems faster than time consuming Monte-Carlo simulations. Furthermore, the theoretical throughput of communication systems can be expressed as  $T = (1 - P_{\text{out}})R$ , if an unlimited number of ARQs is assumed. Using (20) to evaluate  $P_{\text{out}}$ , we plot in Fig. 1 the difference  $\Delta_T$  between the theoretical throughput of a MIMO system and the one of a DMIMO system with two BSs in function of the rate  $R$  and  $\gamma_2(\text{dB})$ , for  $\gamma = 1$ ,  $p = 3$ ,  $q = 4$ , both downlink and uplink cases in the left hand side and right hand side of the figure, respectively. Moreover as indicated in the top right corner of each sub-figures, we set  $\gamma_1 = 1$  and  $\gamma_1 = \gamma_2 + 1$  for the DMIMO system and MIMO system, respectively, in order to fairly compare the two systems for the same

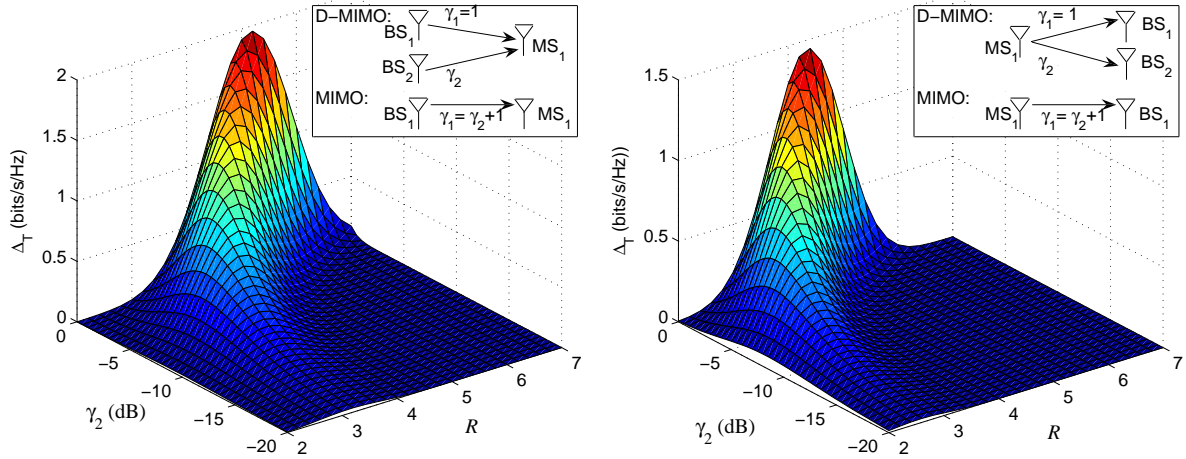


Figure 1: Theoretical throughput comparison of a MIMO system against a DMIMO system with two base stations: Downlink (left), Uplink (right)

total power. Figure 1 shows that a throughput gain resulting from spatial diversity can be achieved by using a DMIMO system instead of a MIMO system. In Fig. 1, maximum gains of 2 and 1.5 bits/s/Hz are reached for  $R \simeq 4.2$  and 3.4 bits/s/Hz in the downlink and uplink cases, respectively, when  $\gamma_2 = 0$  dB.

The formula in (20) can also be utilised in cooperative communication [15] to evaluate the outage probability of the cooperative relay (R) channel, i.e., when both BS and relay station (RS) transmit to the MS. Furthermore, the result in the first line of (19) can be used to evaluate  $\mathbf{E}\{C(\mathbf{H})\}_{\mathbf{H}}$  for any MIMO communication system where its instantaneous capacity is expressed as  $C(\mathbf{H}) = \log_2 |\mathbf{I} + \mathbf{H}\Delta\mathbf{H}^\dagger|$ , with  $\Delta = \text{diag}([\delta_0, \delta_1, \dots, \delta_{N_t}])$  is a  $N_t \times N_t$  diagonal matrix. Using this result a novel amplify-and-forward scheme for cooperative communication based on the expectation of the RS to MS channel has been designed in [16].

## 5. Conclusion

In this paper, the instantaneous capacity of DMIMO systems has been shown to be asymptotically equivalent to a Gaussian RV and an asymptotical approximation of the outage probability for DMIMO systems has been derived for both downlink and uplink cases. Numerical results have indicated that the accuracy of our expression is sufficient for a two-BSs DMIMO systems with two transmit antennas per BS, when  $R \geq 4$  and  $\Delta_\gamma$ (dB) is low. Moreover, the accuracy increases as the rate or the number of nodes and antennas in the DMIMO system increases. Our expression can be used to swiftly evaluate and compare the outage probability of DMIMO systems, and it has been utilised here to show that DMIMO systems can achieve higher throughput than MIMO systems by taking advantage of spatial diversity. Some other applications of our analysis for cooperative communications have also been introduced.

## A Appendix

**Theorem 1 ([17])** Given a square matrix  $\mathbf{A} \in \mathbb{C}^{n \times n}$  and a vector  $\mathbf{x} \in \mathbb{C}^n$ , the improper integral  $I = \int_{\mathbb{C}^n} e^{(-\pi\mathbf{x}^\dagger \mathbf{A}\mathbf{x})} d\mathbf{x}$  exists in the Lebesgue sense if and only if the real part of all the eigenvalues of  $\mathbf{A}$  is positive. Then if  $I$  exists, the following equality holds  $I = \det(\mathbf{A})^{-1}$ .

**Equality 1 ([7])**  $\frac{1}{j2\pi} \int_{\mathcal{D}_{f_0}^j} \int_{\mathcal{D}_{g_0}} \exp(f(-\kappa g + x) + gy) df dg = \exp(xy/\kappa)/\kappa$ , holds for any  $x, y, \kappa, f, g \in \mathbb{C}$ . The domains  $\mathcal{D}_{f_0}^j$  and  $\mathcal{D}_{g_0}$  are defined as  $\mathcal{D}_{f_0}^j \triangleq (f_0 - j\infty, f_0 + j\infty)$  and  $\mathcal{D}_{g_0} \triangleq (g_0 - \infty, g_0 + \infty)$ , respectively.

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